

Seven-Dimensional Super-Yang–Mills Theory in $\mathcal{N} = 1$ Superfields

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Abstract

We give a gauge-covariant formulation of seven-dimensional super-Yang–Mills theory in terms of $\mathcal{N} = 1$ superfields. Furthermore, we show that five and seven dimensions are the only cases where such a formulation exists by analysing the interplay of Lorentz and R symmetries. The action is expressed in terms of field strengths and a Chern–Simons-like superpotential. Each term is manifestly $\mathcal{N} = 1$ supersymmetric, Lorentz invariant in four dimensions and gauge invariant under superfield gauge transformations, including those that do not preserve Wess–Zumino gauge.

1 Introduction

Superfields [1] are a very convenient tool for model building in $\mathcal{N} = 1$ supersymmetric¹ theories. For higher supersymmetries, however, superspace formulations [2] are rather less convenient and have not been used much. This also applies to higher-dimensional theories, which from the four-dimensional perspective correspond to $\mathcal{N} = 2$ or $\mathcal{N} = 4$ supersymmetry. There have been various reformulations of higher-dimensional supersymmetric theories in terms of $\mathcal{N} = 1$ superfields, starting with [3] for the ten-dimensional theory. There was renewed interest in the subject with the advent of higher-dimensional (orbifold) field theory models since around 2000 [4–6]. The general idea is that the fields of any theory with higher supersymmetry still form multiplets under an $\mathcal{N} = 1$ subset of the symmetry, and hence fit into $\mathcal{N} = 1$ superfields in terms of which one can write the action. For example, five- or six-dimensional models correspond to $\mathcal{N} = 2$ supersymmetry, where the vector multiplet gives rise to a vector and a chiral superfield, while the hypermultiplet gives two chiral superfields. In seven to ten dimensions, there is only the vector multiplet (in rigid

¹Throughout this paper, we use the four-dimensional \mathcal{N} , i.e. $\mathcal{N} = 1$ corresponds to four supercharges.

supersymmetry, i.e. excluding supergravity), which leads to one vector and three chiral superfields. This idea has also been extended to five-dimensional supergravity [7] (see also [8] for the linearised case).

It might seem that in this approach one loses a lot of manifest symmetry. However, usually one is interested in $\mathcal{N} = 1$ models in four dimensions anyway, and the remaining supersymmetry is broken by the process of compactification, such as orbifold twists, non-trivial holonomy or intersecting branes. The $\mathcal{N} = 1$ superfield approach then suits for dealing with parts of the model that do not respect the full supersymmetry, but only a subset, such as localised matter.

For gauge theories, however, there is a common problem in these approaches: The action is in general not formulated in terms of covariant derivatives and fields strengths only, but rather contains explicit factors of the vector superfield V and/or partial derivatives in the extra dimensions. This means that the invariance of the action under gauge transformations that do not respect Wess–Zumino (WZ) gauge is not guaranteed and has to be enforced by a Wess–Zumino–Witten (WZW) like term [3] of the form

$$(\partial V) \frac{\sinh L_V - L_V}{L_V^2} (\partial V) = (\partial V) \left(\frac{L_V}{3!} + \frac{L_V^3}{5!} + \dots \right) (\partial V) \quad (1.1)$$

with the Lie bracket $L_V X = [V, X]$. Here ∂V denotes some partial derivatives of V in the internal dimensions. This term vanishes in WZ gauge since already the first term in the series is $\mathcal{O}(V^3)$, but for a general form of V , including a $\theta = \bar{\theta} = 0$ component, the series is indeed infinite.

The situation has been significantly simplified in five dimensions by Hebecker [9] by giving a fully gauge covariant description. A key ingredient was the introduction of a covariant derivative in the extra dimension, which allowed to define a field strength analogous to the standard W_α , in terms of which the action can be easily formulated. The aim of this paper is to extend this formulation to other dimensions. This will, however, turn out to be possible only for the seven-dimensional case, and possibly for six-dimensional $\mathcal{N} = 4$ super-Yang–Mills theory, as can be seen by considering the respective R symmetries.

The paper is organised as follows: In Section 2, we will review the covariant description of [9] and discuss the possibilities of extending the method to other dimensions. In Section 3, we will then discuss the component form of the seven-dimensional theory, and the decomposition in terms of four-dimensional degrees of freedom. The superfield embedding and action are given in Section 4. We will finally conclude in Section 5 and mention some potential applications. Finally, in the appendices we present some details on the reduction of the spinors and the supersymmetry transformations.

2 Covariant Formulations

2.1 Five Dimensions

A covariant formulation of five-dimensional super-Yang–Mills theory was given by Hebecker [9], and we will briefly review the central point. In five dimensions, the off-shell theory is known [10]. It involves the gauge vector A_M , a scalar B and a symplectic

Majorana spinor λ_I as dynamical fields, as well as an $SU(2)_R$ triplet X^i of auxiliary fields². They form an off-shell representation of the SUSY algebra under the transformations

$$\delta A_M = i\bar{\varepsilon}^I \gamma_M \lambda_I, \quad (2.1a)$$

$$\delta B = i\bar{\varepsilon}^I \lambda_I, \quad (2.1b)$$

$$\delta \lambda_I = \gamma^{MN} F_{MN} \varepsilon_I + \gamma^M D_M B \varepsilon_I + iX_i (\sigma^i)_I^J \varepsilon_J, \quad (2.1c)$$

$$\delta X^i = \bar{\varepsilon}^I (\sigma^i)_I^J \gamma^M D_M \lambda_J + i[B, \lambda_J] (\sigma^i)_I^J \bar{\varepsilon}^I. \quad (2.1d)$$

Here also the variation parameter ε_I is a symplectic Majorana spinor, which corresponds to two 4D Weyl spinors. The key observation is now that under a SUSY transformation generated by one of these Weyl spinors, one can identify a vector and a chiral superfield,

$$V = -\theta \sigma^\mu \bar{\theta} A_\mu + i\theta^2 \bar{\theta} \bar{\lambda} - i\bar{\theta}^2 \theta \lambda + \frac{1}{2} \theta^4 (X^3 - D_5 B), \quad (2.2)$$

$$\Phi = A_5 + iB + 2\theta\psi + \theta^2 (X^1 + iX^2). \quad (2.3)$$

Here λ and ψ arise from a suitable decomposition of the 5D spinor λ_I into Weyl spinors. Under gauge transformations with superfield parameter Λ , these superfields transform as

$$e^{2V} \longrightarrow e^{-i\bar{\Lambda}} e^{2V} e^{i\Lambda}, \quad \Phi \longrightarrow e^{-i\Lambda} (\Phi - i\partial_5) e^{i\Lambda}. \quad (2.4)$$

V is given in Wess–Zumino gauge, and consequently, gauge and supersymmetry transformations mix (i.e. a SUSY transformation requires a compensating gauge transformation to return to WZ gauge). The same now applies to the chiral field Φ .

The particular form of Φ allows to define a derivative that is covariant with respect to supersymmetry and gauge symmetry,

$$\nabla = \partial_5 + i\Phi. \quad (2.5)$$

With this derivative, one can define a covariantly transforming “extra-dimensional field strength”

$$Z = e^{-2V} \nabla e^{2V}, \quad Z \longrightarrow e^{-i\Lambda} Z e^{i\Lambda}. \quad (2.6)$$

Z , together with the usual field strength W_α , elegantly reproduces the 5D component Lagrangean:

$$\begin{aligned} \mathcal{L}_5 &= \frac{1}{4} \int d^4\theta \operatorname{tr} Z^2 + \frac{1}{4} \left(\int d^2\theta \operatorname{tr} W^\alpha W_\alpha + \text{H.c.} \right) \\ &= -\frac{1}{4} F_{MN} F^{MN} - \frac{1}{2} D_M B D^M B - \frac{i}{2} \bar{\lambda}^I \Gamma^M D_M \lambda_I + \frac{1}{2} X^i X_i + \frac{1}{2} \bar{\lambda}^I [B, \lambda_I]. \end{aligned} \quad (2.7)$$

The superfield action is constructed from covariant quantities only and does not contain explicit factors of V , so it is manifestly gauge invariant under arbitrary gauge transformations, in particular under those which do not maintain Wess–Zumino gauge.

²For consistency with later sections, our notation differs from [9].

2.2 R Symmetries and General Dimensions

The approach outlined in the previous Section cannot be generalised to arbitrary dimensions. This can be seen from a symmetry argument: When expressed in four-dimensional degrees of freedom, $4 + d$ -dimensional super-Yang–Mills theory corresponds to a $\mathcal{N} = 2$ or $\mathcal{N} = 4$ theory, which would have an R symmetry $SU(2)$ or $SU(4)$, respectively. However, there is extra structure which reduces the R symmetry: Besides the vector field, the theory will contain two or six scalar fields, which separate into d extra-dimensional vector components, which transform under $SO(d)$ and a number of $(4 + d)$ -dimensional scalars which transform under $SO(2 - d)$ or $SO(6 - d)$, respectively. The fermionic sector will likewise contain two or four Weyl fermions obtained from the decomposition of the gaugino, which form a fundamental representation of $SU(2)$ or $SU(4)$. A superfield approach such as above singles out one of the Weyl fermions to be the 4D gaugino, hence it breaks the manifest R symmetry from $SU(2)$ to nothing or $SU(4) \rightarrow SU(3)$, respectively.

Hence, we have different symmetries in the scalar and the fermionic sector, unless $d = 1$, where there is no such symmetry, or $d = 3$, where we can embed the scalars' diagonal $SO(3)$ in the fermions' $SU(3)$. In a more pedestrian view, we want to form chiral multiplets whose scalar components are of the form $A + iB$, where A is an extra-dimensional vector component and B is a true scalar. The number of A 's and B 's coincides only in five and seven dimensions.

This argument does furthermore suggest that a similar description is possible for six-dimensional $\mathcal{N} = 2$ super-Yang–Mills theory, where the fermions again have an $SU(3)$ symmetry into which the geometric $SO(2)$ can be embedded. Then one of the three chiral multiplets will be a pure adjoint matter multiplet.

3 Component Lagrangean

Minimal supersymmetry in seven to ten dimensions has 16 supercharges, i.e. it is $\mathcal{N} = 4$ from the four-dimensional point of view. Hence the only multiplet with spins not greater than one is the Yang-Mills multiplet, but there are no matter multiplets. The precise field content and action in seven dimensions can be derived from the super-Yang–Mills theory in ten dimensions, where the theory has the “minimal” field content, i.e. just the gauge field $A_{\widehat{M}}$ and a Majorana–Weyl spinor Ξ in the adjoint representation of the gauge group G . The Lagrangean in ten dimensions is

$$\mathcal{L}_{10} = -\frac{1}{4}F_{\widehat{M}\widehat{N}}F^{\widehat{M}\widehat{N}} - \frac{i}{2}\Xi\Gamma^{\widehat{M}}D_{\widehat{M}}\Xi, \quad (3.1)$$

the SUSY transformations are

$$\delta A_{\widehat{M}} = \frac{i}{2}\bar{\epsilon}\Gamma_{\widehat{M}}\Xi, \quad \delta\Xi = -\frac{1}{4}F_{\widehat{M}\widehat{N}}\Gamma^{\widehat{M}\widehat{N}}\epsilon. \quad (3.2)$$

Here and in the following, we set the coupling constant to $g = 1$. It can always be restored by dimensional arguments.

The reduction to seven dimensions is basically straightforward. The only subtlety lies in the different types of spinors in ten and seven dimensions [11]: The seven-dimensional

superalgebra has an $SU(2)_R$ symmetry, even for minimal supersymmetry. Some details about the reduction are given in Appendix B. The fields of the seven-dimensional theory are thus a vector A_M which is a singlet under the R symmetry, a triplet of scalars B_i and doublet of spinors Ψ_I which satisfy a symplectic Majorana condition,

$$\Psi_I = \varepsilon_{IJ} C (\bar{\Psi}^J)^T. \quad (3.3)$$

The Lagrangean is

$$\begin{aligned} \mathcal{L}_7 = & -\frac{1}{4} \text{tr} F_{MN} F^{MN} - \frac{1}{2} \text{tr} D_M B_i D^M B^i + \frac{1}{4} \text{tr} [B_i, B_j] [B^i, B^j] \\ & - \frac{i}{2} \text{tr} \bar{\Psi}^I \Gamma^M D_M \Psi_I - \frac{i}{2} \text{tr} \bar{\Psi}^I \left[B_i (\sigma^i)_I{}^J, \Psi_J \right]. \end{aligned} \quad (3.4)$$

It is invariant under the SUSY transformations

$$\delta A_M = \frac{i}{2} \bar{\varepsilon}^I \Gamma_M \Psi_I, \quad (3.5a)$$

$$\delta B_i = \frac{1}{2} \bar{\varepsilon}^I (\sigma_i)_I{}^J \Psi_J, \quad (3.5b)$$

$$\delta \Psi_I = -\frac{1}{4} F_{MN} \Gamma^{MN} \varepsilon_I + \frac{i}{2} \Gamma^M D_M (B_i \sigma^i)_I{}^J \varepsilon_J + \frac{1}{4} \varepsilon^{ijk} [B_i, B_j] (\sigma_k)_I{}^J \varepsilon_J. \quad (3.5c)$$

Here the transformation parameter is again a symplectic Majorana spinor ε_I .

When checking the invariance of the Lagrangeans (3.1) and (3.4), the only pieces which do not cancel immediately are quartic expressions in the fermions, $\sim \Xi \Xi \Xi \varepsilon$ and $\sim \Psi \Psi \Psi \varepsilon$, respectively. These can be seen to vanish using Fierz transformations and the symmetry properties of (symplectic) Majorana spinor products. We collect our conventions regarding seven-dimensional spinors in Appendix A.2.

3.1 Four-Dimensional Degrees of Freedom

We will now reformulate the Lagrangean in terms of four-dimensional degrees of freedom. Thus, the manifest Lorentz symmetry is broken to $SO(1,6) \rightarrow SO(1,3) \times SO(3) \cong SO(1,3) \times SU(2)$, while the R symmetry is untouched. In the bosonic sector, the vector splits into a four-dimensional vector A_μ and a triplet of “gauge scalars” A_i . The $SU(2)_R$ triplet B_i just carries over³.

The fermionic sector requires more work. From the Γ matrices in Appendix A.2, we see that the four-dimensional chirality matrix is

$$\Gamma_* = i\Gamma^0 \Gamma^1 \Gamma^2 \Gamma^3 = \begin{pmatrix} -\mathbb{1} & & & \\ & \mathbb{1} & & \\ & & -\mathbb{1} & \\ & & & \mathbb{1} \end{pmatrix}. \quad (3.6)$$

³The A_i and B_i are triplets under different copies of $SU(2)$. To avoid excessive notation and for later convenience, we use the same indices.

The charge conjugation matrix is

$$B = \Gamma^2 \Gamma^5 = \begin{pmatrix} 0 & 0 & -\epsilon_{\alpha\beta} \\ 0 & \epsilon_{\alpha\beta} & 0 \\ \epsilon^{\dot{\alpha}\dot{\beta}} & 0 & 0 \end{pmatrix}. \quad (3.7)$$

Hence we can decompose the pair of symplectic Majorana gauginos in terms of four Weyl spinors as follows:

$$\Psi_1 = \begin{pmatrix} \lambda_{1\alpha} \\ \bar{\lambda}_2^{\dot{\alpha}} \\ \lambda_{3\alpha} \\ \bar{\lambda}_4^{\dot{\alpha}} \end{pmatrix}, \quad \Psi_2 = \begin{pmatrix} -\lambda_{4\alpha} \\ -\bar{\lambda}_3^{\dot{\alpha}} \\ \lambda_{2\alpha} \\ \bar{\lambda}_1^{\dot{\alpha}} \end{pmatrix}. \quad (3.8)$$

Similarly, the SUSY transformation parameter ε_I can be decomposed into four Weyl spinors ϵ_1 to ϵ_4 and their conjugates. A direct computation shows that under the R symmetry, the λ_r arrange into doublets

$$\begin{pmatrix} \lambda_1 \\ \lambda_4 \end{pmatrix}, \quad \begin{pmatrix} \lambda_2 \\ \lambda_3 \end{pmatrix}, \quad (3.9)$$

while under the geometric $SU(2)$ we have doublets

$$\begin{pmatrix} \lambda_1 \\ \lambda_3 \end{pmatrix}, \quad \begin{pmatrix} \lambda_2 \\ \lambda_4 \end{pmatrix}. \quad (3.10)$$

Thus, the λ_r transform as a $(\mathbf{2}, \mathbf{2}) = \mathbf{4}$ under $SU(2) \times SU(2) \cong SO(4)$, while the scalar triplets $A_i \sim (\mathbf{3}, \mathbf{1})$ and $B_i \sim (\mathbf{1}, \mathbf{3})$ can be represented as (anti)-selfdual two-index tensors of $SO(4)$.

In full dimensional reduction, the theory would obtain an $SU(4)$ R symmetry as enhancement of the $SO(4)$. The spinors simply lift to a $\mathbf{4}$ of $SU(4)$ [12]. The scalars then become a $\mathbf{6}$, satisfying a reality condition (this is consistent because the $\mathbf{6} \sim \square$ is a self-conjugate representation, the condition being $\phi_{ij} = \frac{1}{2}\epsilon_{ijkl}\bar{\phi}^{kl}$). This condition rules out a further enhancement of the R symmetry to $U(4)$. Here, however, the scalar sector prohibits such an enhancement since the A_i and B_i are genuinely different, so indeed $SO(4)$ is the largest admissible R symmetry group.

In Appendix C we have collected the supersymmetry transformations expressed explicitly in terms of four-dimensional quantities. For the superfield formulation, we have to single out one particular transformation parameter, which will break the manifest R symmetry to $SO(3)$ which we identify with the diagonal $SU(2)$. In particular, the fermions decompose as $\mathbf{4} \rightarrow \mathbf{1} \oplus \mathbf{3}$, while the scalars again form two triplets. The diagonal $SU(2)$ ensures gauge covariance, i.e. preservation of the two-triplet structure of the scalars without mixing the A_i and B_i .

From a practical point of view, we require the scalar components of the chiral multiplets to be of the form $\phi_i = A_i + iB_i$ and demand that $\delta\phi_i$ does not depend on $\bar{\epsilon}$. This singles

out the supersymmetry parameter choice $\epsilon_1 = \epsilon_2 = 0$, $\epsilon_3 = \epsilon_4 \equiv \sqrt{2}\epsilon$. With this choice, the following fields transform as chiral multiplets:

$$\phi_1 = A_5 + iB_1, \quad \psi_1 = i(\lambda_1 - \lambda_2), \quad (3.11a)$$

$$\phi_2 = A_6 + iB_2, \quad \psi_2 = -(\lambda_1 + \lambda_2), \quad (3.11b)$$

$$\phi_3 = A_7 + iB_3, \quad \psi_3 = i(\lambda_4 - \lambda_3). \quad (3.11c)$$

Explicitly, their transformation is

$$\delta\phi_i = \sqrt{2}\epsilon\psi_i, \quad \delta\psi_i = -\sqrt{2}i(\partial_\mu\phi_i - \partial_i A_\mu + i[A_\mu, \phi_i])\sigma^\mu\bar{\epsilon} - \sqrt{2}F_i\epsilon. \quad (3.12)$$

The extra terms in $\delta\psi_i$ make the right-hand side gauge covariant: The bracket is just $F_{\mu i} + iD_\mu B_i$. The multiplets still are on-shell, i.e. the auxiliary fields are fixed to be

$$F_i = -\frac{1}{2}\varepsilon_{ijk}(F_{jk} + 2iD_j B_k - i[B_j, B_k]). \quad (3.13)$$

Here the expression in brackets is reminiscent of the field strength of the complex internal gauge field ϕ_i .

The remaining fields A_μ and $\chi = \lambda_3 + \lambda_4$ form a vector multiplet,

$$\delta A_\mu = -\frac{i}{\sqrt{2}}(\epsilon\sigma_\mu\bar{\chi} - \chi\sigma_\mu\bar{\epsilon}), \quad \delta\chi = -\sqrt{2}F_{\mu\nu}\sigma^{\mu\nu}\epsilon + \sqrt{2}iD\epsilon, \quad (3.14)$$

where the auxiliary field is $D = D_i B_i$.

Observe that the supersymmetry and gauge transformations mix: As usual for the vector multiplet, the supersymmetry transformations have to be accompanied by a gauge transformation which reestablishes Wess–Zumino gauge. This implies that they close only up to a gauge transformation,

$$[\delta_\epsilon, \delta_\eta] A_\mu = -2i(\epsilon\sigma^\nu\bar{\eta} - \eta\sigma^\nu\bar{\epsilon})\partial_\nu A_\mu - \delta_{\text{gauge}}. \quad (3.15)$$

Here δ_{gauge} is a transformation with the field-dependent parameter

$$i(\epsilon\sigma^\mu\bar{\eta} - \eta\sigma^\mu\bar{\epsilon})A_\mu. \quad (3.16)$$

Now the same phenomenon appears for the chiral multiplets, which have a gauge condition imposed on them: The real part of the scalar component transforms inhomogeneously while the rest are tensors. This condition is violated by simple admixtures of $\phi_i\epsilon$ to ψ_i , and the violation needs to be compensated by a suitable gauge transformation which leads to (3.16).

4 Superfield Lagrangean

In this section we express the component theory of Section 3 in terms of $\mathcal{N} = 1$ superfields. Our conventions regarding Weyl spinors and van der Waerden notation are given in Appendix A.3. We first specify the embedding of the fields into superfields. This will enable us to define a covariant extra-dimensional derivative, which in turn is crucial to formulate the Lagrangean.

4.1 Field Embedding

We embed the chiral multiplets (3.11) into chiral superfields Φ_i , while the vector multiplet (3.14) forms a vector superfield in WZ gauge,

$$V = -\theta\sigma^\mu\bar{\theta}A_\mu + \frac{1}{\sqrt{2}}\theta^2\bar{\theta}\bar{\chi} + \frac{1}{\sqrt{2}}\bar{\theta}^2\theta\chi + \frac{1}{2}\theta^4D, \quad (4.1)$$

$$\Phi_i = \phi_i + \sqrt{2}i\theta\psi_i + \theta^2F_i. \quad (4.2)$$

A gauge transformation now takes a complete chiral multiplet Λ as parameter. (WZ gauge is preserved for Λ having only a real scalar component.) The presence of the vector field components in ϕ_i implies that the superfields transform as

$$\Phi_i \longrightarrow e^{-i\Lambda}(\Phi_i - i\partial_i)e^{i\Lambda}, \quad e^{2V} \longrightarrow e^{-i\bar{\Lambda}}e^{2V}e^{i\Lambda}. \quad (4.3a)$$

We define a supersymmetric covariant derivative in the extra dimensions,

$$\nabla_i = \partial_i + i\Phi_i, \quad (4.4)$$

which transforms as $\nabla_i \rightarrow e^{-i\Lambda}\nabla_ie^{i\Lambda}$. Here it is implied that Φ_i acts according to the representation of the field it is applied to. In particular, we have

$$\nabla_ie^{2V} = \partial_ie^{2V} + i\bar{\Phi}_ie^{2V} - ie^{2V}\Phi_i \quad (4.5)$$

for the adjoint vector superfield.

4.2 Lagrangean

The Lagrangean contains three pieces: The usual gauge kinetic term, a kinetic term for the chiral superfields and a superpotential-like term. All three are by themselves invariant under $\mathcal{N} = 1$ supersymmetry and gauge symmetry. Their relative coefficients are determined by higher-dimensional Lorentz invariance.

There are two field-strength-like superfields that can be constructed from V and Φ . The first is the usual chiral field strength $W_\alpha = -\frac{1}{4}\bar{D}^2e^{-2V}D_\alpha e^{2V}$, which gives rise to the action

$$\frac{1}{16} \int d^2\theta W^\alpha W_\alpha + \text{H.c.} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{i}{2}\chi\sigma^\mu D_\mu\bar{\chi} + \frac{1}{2}D^2. \quad (4.6)$$

The second piece contains extra-dimensional derivatives acting on V . Equation (4.5) implies that the simplest covariantly transforming object is [9]

$$Z_i = e^{-2V}\nabla_ie^{2V}. \quad (4.7)$$

It is neither chiral nor real. Under gauge transformations, this field transforms as $Z_i \rightarrow e^{-i\Lambda}Z_ie^{i\Lambda}$. Note that this implies that the components of Z_i are not tensors even under

WZ gauge preserving transformations (unless the gauge group is Abelian, in which case Z_i is gauge invariant), for which the parameter is (in a non-chiral superfield representation)

$$\Lambda = \lambda + i\theta\sigma^\mu\bar{\theta}\partial_\mu\lambda + \frac{1}{4}\theta^4\Box\lambda, \quad (4.8)$$

where λ is real. Here the variation of Z_i is

$$\delta Z_i = i[Z_i, \Lambda] = i[Z_i, \lambda] + i\left[Z_i, i\theta\sigma^\mu\bar{\theta}\partial_\mu\lambda + \frac{1}{4}\theta^4\Box\lambda\right]. \quad (4.9)$$

The second term shows that the higher components ($\theta\bar{\theta}$ and higher) transform inhomogeneously. Z_i is not Hermitean, but satisfies $Z_i^\dagger = e^{2V}Z_ie^{-2V}$, such that the lowest-dimensional gauge-invariant term that can be formed, $\text{tr } Z_i Z_i$ is Hermitean⁴. This is the second piece of the action,

$$\begin{aligned} \frac{1}{4} \int d^4\theta \text{tr } Z_i Z_i &= -\frac{1}{2}F_{\mu i}F^{\mu i} - \frac{1}{2}D_\mu B_i D^\mu B_i - DD_i B_i + 2F_i \bar{F}_i \\ &\quad - \frac{i}{2}\psi_i \sigma^\mu D_\mu \bar{\psi}_i - \frac{1}{2}\chi D_i \psi_i - \frac{1}{2}\bar{\chi} D_i \bar{\psi}_i \\ &\quad + \frac{1}{2}\chi [B_i, \psi_i] - \frac{1}{2}\bar{\chi} [B_i, \bar{\psi}_i]. \end{aligned} \quad (4.10)$$

The final piece contributes $F_{ij}F^{ij}$ and related terms via the auxiliary fields F_i . It is given as a Chern–Simons-like superpotential contribution,

$$W = \varepsilon_{ijk}\Phi_i \left(\partial_j \Phi_k + \frac{i}{3} [\Phi_j, \Phi_k] \right), \quad (4.11)$$

which gives rise to

$$\begin{aligned} \frac{1}{4} \int d^2\theta W + \text{H.c.} &= \frac{1}{4}\varepsilon_{ijk}F_i (F_{jk} + 2iD_j B_k - i[B_j, B_k]) \\ &\quad + \frac{1}{4}\varepsilon_{ijk}\psi_i D_j \psi_k - \frac{1}{4}\varepsilon_{ijk}\psi_i [B_j, \psi_k] + \text{H.c.} \end{aligned} \quad (4.12)$$

The left-hand side is not obviously gauge invariant. Rather, the superpotential transforms as

$$\delta W = \frac{1}{3}\varepsilon_{ijk} (e^{-i\Lambda}\partial_i e^{i\Lambda}) (e^{-i\Lambda}\partial_j e^{i\Lambda}) (e^{-i\Lambda}\partial_k e^{i\Lambda}). \quad (4.13)$$

However, we will now argue that the action is still gauge invariant: First note that δW depends only on the gauge transformation parameter Λ , but not on the fields V or Φ_i . Furthermore, δW vanishes under the $d^2\theta$ integral if Λ contains only a scalar component, which includes WZ preserving gauge transformations, but also transformations which endow V with a scalar ($\theta = \bar{\theta} = 0$) component.

⁴Alternatively, one could define a real field $\tilde{Z}_i = e^V Z_i e^{-V}$, for which $\text{tr } \tilde{Z}_i \tilde{Z}_i = \text{tr } Z_i Z_i$.

Second, we recognise δW as the “winding number density” [13], so the integral $\int d^3y \delta W = \text{const} \cdot n$ gives the winding number of the gauge transformation⁵. It is characterised by the third homotopy group, which is $\pi_3(G) = \mathbb{Z}$ for all compact simple groups.

Third, the integral $\int d^3y \delta W = z + \theta \zeta + \theta^2 F_Z$ is a bona fide chiral superfield in four dimensions, since the x and θ dependence is untouched by the internal-space integral. Hence, under supersymmetry transformations its scalar component should transform as $\delta z = \epsilon \zeta$. However, this transformation together with the quantisation condition $z = \text{const} \cdot n$ implies that $\zeta = 0$, and thus the transformation $\delta \zeta \sim F_Z \epsilon$ in turn requires $F_Z = 0$. Hence $\int d^2\theta \delta W = F_Z = 0$, and the action is gauge invariant for any gauge transformation parameter Λ .

Altogether, we have the following Lagrangean:

$$\begin{aligned}
\mathcal{L}_{\text{SF}} &= \frac{1}{4} \int d^4\theta \text{tr} Z_i Z_i \\
&+ \left[\frac{1}{16} \int d^2\theta W^\alpha W_\alpha + \frac{1}{4} \int d^2\theta \varepsilon_{ijk} \Phi_i \left(\partial_j \Phi_k + \frac{i}{3} [\Phi_j, \Phi_k] \right) + \text{H.c.} \right] \\
&= -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2} F_{\mu i} F^{\mu i} - \frac{1}{2} D_\mu B_i D^\mu B_i + \frac{1}{2} D^2 - D D_i B_i \\
&+ \frac{1}{2} F_i \bar{F}_i + \left[\frac{1}{4} F_i \varepsilon_{ijk} (F_{jk} + 2i D_j B_k - i [B_j, B_k]) + \text{H.c.} \right] \\
&- \frac{i}{2} \chi \sigma^\mu D_\mu \bar{\chi} - \frac{i}{2} \psi_i \sigma^\mu D_\mu \bar{\psi}_i \\
&- \frac{1}{2} \left[\chi (D_i \psi_i - [B_i, \psi_i]) - \frac{1}{2} \varepsilon_{ijk} \psi_i (D_j \psi_k - [B_j, \psi_k]) + \text{H.c.} \right].
\end{aligned} \tag{4.14}$$

Eliminating the auxiliary fields by their equations of motion,

$$F_i = -\frac{1}{2} \varepsilon_{ijk} (F_{jk} + 2i D_j B_k - i [B_j, B_k]), \tag{4.15a}$$

$$D = D_i B_i, \tag{4.15b}$$

we obtain the final expression

$$\begin{aligned}
\mathcal{L}_{\text{SF}} &= -\frac{1}{4} F_{MN} F^{MN} - \frac{1}{2} D_M B_i D^M B_i + \frac{1}{4} [B_i, B_j] [B_i, B_j] \\
&- \frac{i}{2} \chi \sigma^\mu D_\mu \bar{\chi} - \frac{i}{2} \psi_i \sigma^\mu D_\mu \bar{\psi}_i \\
&- \frac{1}{2} \left[\chi (D_i \psi_i - [B_i, \psi_i]) - \frac{1}{2} \varepsilon_{ijk} \psi_i (D_j \psi_k - [B_j, \psi_k]) + \text{H.c.} \right].
\end{aligned} \tag{4.16}$$

This reproduces the original Lagrangean (3.4) when expressed in four-dimensional degrees of freedom. Note also that the auxiliary field expressions (4.15) and (3.13) match.

⁵This is the winding number around the internal space in the phenomenologically interesting case of three compact dimensions. For noncompact y directions, one has to assume suitable boundary conditions for the transformation as $|y| \rightarrow \infty$.

All three pieces of the Lagrangean are by themselves 4D Lorentz invariant, $\mathcal{N} = 1$ supersymmetric and gauge invariant. Their relative coefficients are fixed by the requirement of seven-dimensional Lorentz symmetry, which, together with the manifest supersymmetry, enforces $\mathcal{N} = 4$ supersymmetry. Note that, in particular, the action does not contain explicit factors of V and consequently does not require a Wess–Zumino–Witten-like term [3, 4] to ensure gauge invariance under transformations not preserving WZ gauge.

5 Conclusions and Outlook

For SUSY model building it is rather convenient to have a simple superfield formulation. However, for higher-dimensional supersymmetric gauge theories, these formulations are often rather cumbersome, because of the nontrivial interplay of supersymmetry and gauge symmetry. In this paper we have shown that the simple covariant formulation of [9] cannot be generalised to arbitrary dimensions, but only to the case of $D = 7$. The origin of this fact is the combination of extra-dimensional Lorentz and R symmetries, which requires an equal number of scalars and higher-dimensional vector components to form chiral multiplets.

Furthermore, we have presented a gauge covariant superfield description of seven-dimensional super-Yang–Mills theory. The action contains the usual gauge kinetic term, a Kähler potential term and a superpotential. All three terms are by themselves $\mathcal{N} = 1$ supersymmetric, 4D Lorentz invariant and invariant under superfield gauge transformations, including those that do not preserve WZ gauge. In particular, the vector superfield does not appear explicitly, and hence there is no need for a WZW-like term. The R symmetry argument alluded to above does additionally suggest that the six-dimensional $\mathcal{N} = 4$ (maximal) super-Yang–Mills theory has a similar formulation, which can be obtained from the seven-dimensional case by reduction.

Seven-dimensional field theories can be studied in their own right. However, they also naturally appear in the context of type IIA string theory with D6 branes compactified on a Calabi–Yau, or of M-theory on G_2 manifolds with ADE singularities. The formalism presented here finds a natural application in this setup. Since $\mathcal{N} = 1$ supersymmetry is manifest, and the coordinates are naturally split between the internal and Minkowski space, one can easily treat intersecting branes, which lead to a (possibly spontaneously broken) $\mathcal{N} = 1$ supersymmetric theory in four dimensions. The coupling of localised matter on the intersection to the gauge fields on the brane are directly apparent. Furthermore, the formalism allows for a systematic study of higher-dimensional operators in a supersymmetric fashion.

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A Spinor Conventions

A.1 Generalities

We use the metric $\eta = \text{diag}(-, +, \dots, +)$. The Γ matrices satisfy the algebra $\{\Gamma^M, \Gamma^N\} = 2\eta^{MN}\mathbb{1}$. Hence, Γ^0 is anti-Hermitian, while the rest is Hermitian.

We denote seven-dimensional indices by $M, N, \dots = 0, 1, 2, 3, 4, 5, 6$ and four-dimensional ones by $\mu, \nu, \dots = 0, 1, 2, 3$. Indices in the extra dimensions are denoted by $i, j, \dots = 1, 2, 3$ and are raised and lowered with δ_{ij} .

A.2 Seven-Dimensional Spinors

For the reduction to four dimensions, we use the following explicit representation:

$$\Gamma^\mu = \mathbb{1} \otimes \gamma^\mu, \quad \Gamma^{3+i} = \sigma^i \otimes \gamma^5, \quad (\text{A.1})$$

where the 4×4 matrices γ^μ are given by

$$\gamma^0 = \begin{pmatrix} 0 & \mathbb{1} \\ -\mathbb{1} & 0 \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ \sigma^i & 0 \end{pmatrix}, \quad (\text{A.2})$$

and the σ^i are the Pauli matrices. γ^5 is the product

$$\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} -\mathbb{1} & 0 \\ 0 & \mathbb{1} \end{pmatrix}. \quad (\text{A.3})$$

The gaugino satisfies a symplectic Majorana condition. It can be phrased in terms of the charge conjugation matrix C which generates transpositions, or in terms of $B = -\Gamma^0 C$ which generates complex conjugation,

$$C\Gamma_M C^{-1} = -\Gamma_M^T, \quad B\Gamma_M B^{-1} = \Gamma_M^*, \quad (\text{A.4})$$

and reads in four equivalent formulations (in seven dimensions $BB^* = -\mathbb{1}$)

$$\Psi_I = \varepsilon_{IJ} C (\bar{\Psi}^T)^J, \quad \bar{\Psi}^I = -\varepsilon^{IJ} \Psi_J^T C, \quad (\text{A.5})$$

$$\Psi_I = \varepsilon_{IJ} B (\Psi^*)^J, \quad (\Psi^*)^J = \varepsilon^{IJ} B^* \Psi_J. \quad (\text{A.6})$$

Explicitly, we have

$$C = \Gamma^0 \Gamma^2 \Gamma^5 = \begin{pmatrix} 0 & \varepsilon & 0 \\ -\varepsilon & 0 & \varepsilon \\ 0 & -\varepsilon & 0 \end{pmatrix}, \quad B = \Gamma^2 \Gamma^5 = \begin{pmatrix} 0 & 0 & -\varepsilon \\ \varepsilon & 0 & 0 \\ -\varepsilon & 0 & 0 \end{pmatrix}. \quad (\text{A.7})$$

A.3 Weyl Spinors

Our conventions regarding four-dimensional Weyl spinors are similar to [14], mainly differing by the different choice of metric. We denote left-handed (right-handed) Weyl spinors by undotted (dotted) indices. They are raised and lowered with the ϵ symbol in the following way:

$$\psi^\alpha = \epsilon^{\alpha\beta} \psi_\beta, \quad \psi_\alpha = \psi^\beta \epsilon_{\beta\alpha}, \quad (\text{A.8})$$

$$\bar{\chi}^{\dot{\alpha}} = \bar{\chi}_{\dot{\beta}} \epsilon^{\dot{\beta}\dot{\alpha}}, \quad \bar{\chi}_{\dot{\alpha}} = \epsilon_{\dot{\alpha}\dot{\beta}} \bar{\chi}^{\dot{\beta}}, \quad (\text{A.9})$$

where ϵ is defined as

$$\epsilon_{12} = \epsilon^{12} = 1, \quad \epsilon_{\dot{1}\dot{2}} = \epsilon^{\dot{1}\dot{2}} = -1. \quad (\text{A.10})$$

Note that this convention implies that $\epsilon^{\alpha\beta} \epsilon_{\beta\gamma} = -\delta_\gamma^\alpha$. For spinor products, undotted indices are contracted top-down, dotted ones bottom up,

$$\psi\chi = \psi^\alpha \chi_\alpha = \chi\psi, \quad \bar{\psi}\bar{\chi} = \bar{\psi}_{\dot{\alpha}} \bar{\chi}^{\dot{\alpha}} = \bar{\chi}\bar{\psi}. \quad (\text{A.11})$$

Hermitean conjugation turns undotted into dotted indices and vice versa,

$$(\psi_\alpha)^\dagger = \bar{\psi}_{\dot{\alpha}}, \quad (\psi^\alpha)^\dagger = \bar{\psi}^{\dot{\alpha}}, \quad \Rightarrow \quad (\psi^\alpha \chi_\alpha)^\dagger = \bar{\chi}_{\dot{\alpha}} \bar{\psi}^{\dot{\alpha}} = \bar{\chi}\bar{\psi}. \quad (\text{A.12})$$

We define two sets of σ matrices, $\sigma_{\alpha\dot{\alpha}}^\mu = (\mathbb{1}, \sigma^i)$ and $(\tilde{\sigma}^\mu)^{\dot{\alpha}\alpha} = (-\mathbb{1}, \sigma^i)$. They are related by raising indices, however, there is an extra minus sign,

$$\epsilon^{\alpha\beta} \sigma_{\beta\dot{\beta}}^\mu \epsilon^{\dot{\beta}\dot{\alpha}} = -(\tilde{\sigma}^\mu)^{\dot{\alpha}\alpha}, \quad \epsilon_{\dot{\alpha}\dot{\beta}} (\tilde{\sigma}^\mu)^{\dot{\beta}\beta} \epsilon_{\beta\alpha} = -\sigma_{\alpha\dot{\alpha}}^\mu. \quad (\text{A.13})$$

These conventions impose the following index structure on the Γ s and products thereof, repeating in each 4×4 block:

$$\Gamma^{M\dots P} = \begin{pmatrix} A_\alpha{}^\beta & B_{\alpha\dot{\beta}} & * \\ C^{\dot{\alpha}\beta} & D^{\dot{\alpha}}{}_{\dot{\beta}} & * \\ * & * & * \end{pmatrix}. \quad (\text{A.14})$$

However, this does not apply to the charge conjugation matrix and the Γ^0 used in defining $\bar{\Psi} = \Psi^\dagger \Gamma^0$, as these are intertwiners between different representations. In particular, we have

$$B = \Gamma^2 \Gamma^5 = \begin{pmatrix} 0 & 0 & -\epsilon_{\alpha\beta} \\ 0 & \epsilon_{\alpha\beta} & 0 \\ \epsilon^{\dot{\alpha}\dot{\beta}} & 0 & 0 \end{pmatrix}. \quad (\text{A.15})$$

Furthermore, note that the ϵ symbols act in the wrong way (i.e. undotted bottom-up, dotted top-down), so in defining symplectic Majorana spinors there is another minus sign.

Taking this together with the four-dimensional chirality matrix

$$\Gamma_* = i\Gamma^0\Gamma^1\Gamma^2\Gamma^3 = \begin{pmatrix} -\mathbb{1} & & & \\ & \mathbb{1} & & \\ & & -\mathbb{1} & \\ & & & \mathbb{1} \end{pmatrix}, \quad (\text{A.16})$$

we see that a pair of seven-dimensional symplectic Majorana spinors decomposes into left- and right handed Weyl spinors λ_a as

$$\Psi_1 = \begin{pmatrix} \lambda_{1\alpha} \\ \bar{\lambda}_2^{\dot{\alpha}} \\ \lambda_{3\alpha} \\ \bar{\lambda}_4^{\dot{\alpha}} \end{pmatrix}, \quad \Psi_2 = \begin{pmatrix} -\lambda_{4\alpha} \\ -\bar{\lambda}_3^{\dot{\alpha}} \\ \lambda_{2\alpha} \\ \bar{\lambda}_1^{\dot{\alpha}} \end{pmatrix}. \quad (\text{A.17})$$

For convenience, we also list the barred versions:

$$\bar{\Psi}_1 = (-\lambda_2^\alpha, \bar{\lambda}_{1\dot{\alpha}}, -\lambda_4^\alpha, \bar{\lambda}_{3\dot{\alpha}}), \quad \bar{\Psi}_2 = (\lambda_3^\alpha, -\bar{\lambda}_{4\dot{\alpha}}, -\lambda_1^\alpha, \bar{\lambda}_{2\dot{\alpha}}). \quad (\text{A.18})$$

B Reduction from Ten to Seven Dimensions

In this section we detail the reduction of the ten-dimensional Majorana–Weyl spinor Ξ to a seven-dimensional symplectic Majorana spinor. For the purpose of this Appendix, we denote ten-dimensional indices by $\widehat{M}, \widehat{N} = 0, \dots, 9$, and ten-dimensional quantities by hats.

The appearance of the symplectic reality condition can be understood as follows: As a Weyl spinor, Ξ transforms in the **16** of $SO(1, 9)$. (Recall that $SO(1, 9)$ has two spinor representations, **16** and **16'**, which are self-conjugate under charge conjugation.) Hence one can impose an additional Majorana condition, $\Xi^c = \Xi$, to reduce the number of real degrees of freedom to 16. In the reduction to seven dimensions, i.e. in $SO(1, 9) \rightarrow SO(1, 6) \times SO(3) \cong SO(1, 6) \times SU(2)$, the spinor decomposes as **16** \rightarrow (**8**, **2**). Here the **8** is the spinor representation of $SO(1, 6)$. The Majorana condition in ten dimensions translates into a symplectic condition acting on the $SU(2)$ doublet index.

To make this explicit, assume a set Γ^M , $M = 0, \dots, 6$, of seven-dimensional (but 8×8) Γ matrices. Then a convenient set of Γ matrices in ten dimensions is given by

$$\begin{aligned} \widehat{\Gamma}^M &= \sigma_3 \otimes \sigma_3 \otimes \Gamma^M, & \widehat{\Gamma}^7 &= \sigma_2 \otimes \mathbb{1} \otimes \mathbb{1}_8, \\ \widehat{\Gamma}^8 &= -\sigma_1 \otimes \mathbb{1} \otimes \mathbb{1}_8, & \widehat{\Gamma}^9 &= \sigma_3 \otimes \sigma_1 \otimes \mathbb{1}_8. \end{aligned} \quad (\text{B.1})$$

Here $\mathbb{1}$ without subscript denotes the 2×2 unit matrix. This set is convenient because the decomposition of the 32-component ten-dimensional spinor into eight-component seven-dimensional spinors is directly apparent. Furthermore, the relation (B.9) allows for easy identification of the ten-dimensional fields with the seven-dimensional ones.

Let us now impose the Majorana–Weyl nature of the gaugino. The chirality matrix is

$$\widehat{\Gamma}_* = \widehat{\Gamma}^0 \dots \widehat{\Gamma}^9 = \sigma_3 \otimes \sigma_2 \otimes \mathbb{1}_8. \quad (\text{B.2})$$

Here we assumed that $\Gamma^0 \dots \Gamma^6 = -\mathbb{1}_8$, as happens for our choice (A.1). The only other possibility is $\Gamma^0 \dots \Gamma^6 = +\mathbb{1}_8$, in which case we exchange what we call left- and right-handed spinors. The chirality condition $\widehat{\Gamma}_* \Xi = -\Xi$ then implies that Ξ is of the form

$$\Xi = \begin{pmatrix} i\xi_1 \\ \xi_1 \\ i\xi_2 \\ -\xi_2 \end{pmatrix}, \quad (\text{B.3})$$

with – so far – unconstrained eight-component spinors $\xi_{1,2}$.

The Majorana constraint involves the intertwiner with the transposed representation. Let C be this matrix in seven dimensions (Eq. (A.7) for our explicit case), such that $C\Gamma_M C^{-1} = -\Gamma_M^T$. In seven dimensions, this C is symmetric and real⁶, i.e. $C = C^T = C^*$, such that $C = C^{-1}$. Then in ten dimensions, this intertwiner is

$$\widehat{C} = \sigma_1 \otimes \sigma_3 \otimes C, \quad (\text{B.4})$$

which is again real and symmetric and satisfies

$$\widehat{C} \widehat{\Gamma}_M \widehat{C}^{-1} = \widehat{\Gamma}_M^T. \quad (\text{B.5})$$

The Majorana condition is thus

$$\Xi = \begin{pmatrix} i\xi_1 \\ \xi_1 \\ i\xi_2 \\ -\xi_2 \end{pmatrix} = \Xi^c = \widehat{C} \overline{\Xi}^T = \begin{pmatrix} iC\overline{\xi}_2^T \\ C\overline{\xi}_2^T \\ -iC\overline{\xi}_1^T \\ C\overline{\xi}_1^T \end{pmatrix}, \quad (\text{B.6})$$

where we can directly identify the symplectic Majorana condition on the eight-component spinors,

$$\xi_1 = C\overline{\xi}_2^T, \quad \xi_2 = -C\overline{\xi}_1^T. \quad (\text{B.7})$$

Note that Ξ^c is still left-handed. One can check that in ten dimensions, $(\Xi^c)^c = \Xi$. Furthermore, note that indeed the Lorentz generators Σ^{78} , Σ^{89} and Σ^{97} generate an $SU(2)$ rotating ξ_1 and ξ_2 into each other (since $\widehat{\Gamma}^{7,8,9}$ act as the unit matrix on the eight-component spinors):

$$\begin{aligned} \Sigma^{89} : \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} &\longrightarrow -\frac{1}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}, & \Sigma^{97} : \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} &\longrightarrow -\frac{1}{2} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}, \\ \Sigma^{78} : \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} &\longrightarrow -\frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}. \end{aligned} \quad (\text{B.8})$$

⁶The reality and normalisation involve a choice of prefactor, since for any nonzero complex α , αC intertwines just as well as C . However, apart from this rescaling, C is unique.

To identify the seven-dimensional fields, observe that the $\widehat{\Gamma}$'s have been chosen such that the seven-dimensional Γ matrices are already there, so the kinetic term directly descends. Furthermore, we have

$$\bar{\Xi} \widehat{\Gamma}^{7,8,9} \Xi = -2i \bar{\xi}_I \sigma_{IJ}^{1,2,3} \xi_J. \quad (\text{B.9})$$

We thus recover the Lagrangean (3.4) with the identifications

$$A_M = \widehat{A}_M, \quad B_i = \widehat{A}_{6+i}, \quad \Psi_I = \sqrt{2} \xi_I. \quad (\text{B.10})$$

C SUSY Transformations

Here we list the SUSY variations, Eqns. (3.5), expressed in terms of four-dimensional degrees of freedom. In particular, we decompose the gaugino Ψ_I and the transformation parameter ε_I into Weyl spinors λ_a , ϵ_a , $a = 1, \dots, 4$, according to Eq. (3.8). Then we get for the bosons:

$$\begin{aligned} \delta A_\mu = \frac{i}{2} & \left[\bar{\epsilon}_1 \tilde{\sigma}_\mu \lambda_1 - \epsilon_1 \sigma_\mu \bar{\lambda}_1 + \bar{\epsilon}_2 \tilde{\sigma}_\mu \lambda_2 - \epsilon_2 \sigma_\mu \bar{\lambda}_2 \right. \\ & \left. + \bar{\epsilon}_3 \tilde{\sigma}_\mu \lambda_3 - \epsilon_3 \sigma_\mu \bar{\lambda}_3 + \bar{\epsilon}_4 \tilde{\sigma}_\mu \lambda_4 - \epsilon_4 \sigma_\mu \bar{\lambda}_4 \right], \end{aligned} \quad (\text{C.1a})$$

$$\delta A_4 = \frac{i}{2} \left[\bar{\epsilon}_1 \bar{\lambda}_4 - \epsilon_1 \lambda_4 - \bar{\epsilon}_2 \bar{\lambda}_3 + \epsilon_2 \lambda_3 + \bar{\epsilon}_3 \bar{\lambda}_2 - \epsilon_3 \lambda_2 - \bar{\epsilon}_4 \bar{\lambda}_1 + \epsilon_4 \lambda_1 \right], \quad (\text{C.1b})$$

$$\delta A_5 = \frac{1}{2} \left[\bar{\epsilon}_1 \bar{\lambda}_4 + \epsilon_1 \lambda_4 + \bar{\epsilon}_2 \bar{\lambda}_3 + \epsilon_2 \lambda_3 - \bar{\epsilon}_3 \bar{\lambda}_2 - \epsilon_3 \lambda_2 - \bar{\epsilon}_4 \bar{\lambda}_1 - \epsilon_4 \lambda_1 \right], \quad (\text{C.1c})$$

$$\delta A_6 = \frac{i}{2} \left[\bar{\epsilon}_1 \bar{\lambda}_2 - \epsilon_1 \lambda_2 - \bar{\epsilon}_2 \bar{\lambda}_1 + \epsilon_2 \lambda_1 - \bar{\epsilon}_3 \bar{\lambda}_4 + \epsilon_3 \lambda_4 + \bar{\epsilon}_4 \bar{\lambda}_3 - \epsilon_4 \lambda_3 \right], \quad (\text{C.1d})$$

$$\delta B_1 = \frac{1}{2} \left[-\bar{\epsilon}_1 \bar{\lambda}_3 - \epsilon_1 \lambda_3 + \bar{\epsilon}_2 \bar{\lambda}_4 + \epsilon_2 \lambda_4 + \bar{\epsilon}_3 \bar{\lambda}_1 + \epsilon_3 \lambda_1 - \bar{\epsilon}_4 \bar{\lambda}_2 - \epsilon_4 \lambda_2 \right], \quad (\text{C.1e})$$

$$\delta B_2 = \frac{i}{2} \left[\bar{\epsilon}_1 \bar{\lambda}_3 - \epsilon_1 \lambda_3 + \bar{\epsilon}_2 \bar{\lambda}_4 - \epsilon_2 \lambda_4 - \bar{\epsilon}_3 \bar{\lambda}_1 + \epsilon_3 \lambda_1 - \bar{\epsilon}_4 \bar{\lambda}_2 + \epsilon_4 \lambda_2 \right], \quad (\text{C.1f})$$

$$\delta B_3 = \frac{1}{2} \left[\bar{\epsilon}_1 \bar{\lambda}_2 + \epsilon_1 \lambda_2 - \bar{\epsilon}_2 \bar{\lambda}_1 - \epsilon_2 \lambda_1 + \bar{\epsilon}_3 \bar{\lambda}_4 + \epsilon_3 \lambda_4 - \bar{\epsilon}_4 \bar{\lambda}_3 - \epsilon_4 \lambda_3 \right]. \quad (\text{C.1g})$$

For the fermions we obtain

$$\begin{aligned}
\delta\lambda_1 = & -\frac{1}{2}(F_{\mu\nu}\sigma^{\mu\nu} + iF_{45})\epsilon_1 - \frac{1}{2}F_{\mu 6}\sigma^\mu\bar{\epsilon}_2 - \frac{1}{2}(F_{64} - iF_{65})\epsilon_3 - \frac{1}{2}(F_{\mu 4} - iF_{\mu 5})\sigma^\mu\bar{\epsilon}_4 \\
& - \frac{i}{2}D_\mu(B_1 - iB_2)\sigma^\mu\bar{\epsilon}_3 + \frac{i}{2}D_\mu B_3\sigma^\mu\bar{\epsilon}_2 - \frac{i}{2}(D_4 - iD_5)(B_1 - iB_2)\epsilon_2 \\
& - \frac{i}{2}(D_4 - iD_5)B_3\epsilon_3 + \frac{i}{2}D_6(B_1 - iB_2)\epsilon_4 - \frac{i}{2}D_6B_3\epsilon_1 \\
& + \frac{1}{2}[B_1, B_2]\epsilon_1 - \frac{i}{2}[B_1 - iB_2, B_3]\epsilon_4,
\end{aligned} \tag{C.2a}$$

$$\begin{aligned}
\delta\lambda_2 = & -\frac{1}{2}(F_{\mu\nu}\sigma^{\mu\nu} - iF_{45})\epsilon_2 + \frac{1}{2}F_{\mu 6}\sigma^\mu\bar{\epsilon}_1 - \frac{1}{2}(F_{64} + iF_{65})\epsilon_4 + \frac{1}{2}(F_{\mu 4} + iF_{\mu 5})\sigma^\mu\bar{\epsilon}_3 \\
& + \frac{i}{2}D_\mu(B_1 + iB_2)\sigma^\mu\bar{\epsilon}_4 - \frac{i}{2}D_\mu B_3\sigma^\mu\bar{\epsilon}_1 - \frac{i}{2}(D_4 + iD_5)(B_1 + iB_2)\epsilon_1 \\
& - \frac{i}{2}(D_4 + iD_5)B_3\epsilon_4 + \frac{i}{2}D_6(B_1 + iB_2)\epsilon_3 - \frac{i}{2}D_6B_3\epsilon_2 \\
& - \frac{1}{2}[B_1, B_2]\epsilon_2 - \frac{i}{2}[B_1 + iB_2, B_3]\epsilon_3,
\end{aligned} \tag{C.2b}$$

$$\begin{aligned}
\delta\lambda_3 = & -\frac{1}{2}(F_{\mu\nu}\sigma^{\mu\nu} - iF_{45})\epsilon_3 + \frac{1}{2}F_{\mu 6}\sigma^\mu\bar{\epsilon}_4 + \frac{1}{2}(F_{64} + iF_{65})\epsilon_1 - \frac{1}{2}(F_{\mu 4} + iF_{\mu 5})\sigma^\mu\bar{\epsilon}_2 \\
& + \frac{i}{2}D_\mu(B_1 - iB_2)\sigma^\mu\bar{\epsilon}_1 + \frac{i}{2}D_\mu B_3\sigma^\mu\bar{\epsilon}_4 + \frac{i}{2}(D_4 + iD_5)(B_1 - iB_2)\epsilon_4 \\
& - \frac{i}{2}(D_4 + iD_5)B_3\epsilon_1 + \frac{i}{2}D_6(B_1 - iB_2)\epsilon_2 + \frac{i}{2}D_6B_3\epsilon_3 \\
& + \frac{1}{2}[B_1, B_2]\epsilon_3 + \frac{i}{2}[B_1 - iB_2, B_3]\epsilon_2,
\end{aligned} \tag{C.2c}$$

$$\begin{aligned}
\delta\lambda_4 = & -\frac{1}{2}(F_{\mu\nu}\sigma^{\mu\nu} + iF_{45})\epsilon_4 - \frac{1}{2}F_{\mu 6}\sigma^\mu\bar{\epsilon}_3 + \frac{1}{2}(F_{64} - iF_{65})\epsilon_2 + \frac{1}{2}(F_{\mu 4} - iF_{\mu 5})\sigma^\mu\bar{\epsilon}_1 \\
& - \frac{i}{2}D_\mu(B_1 + iB_2)\sigma^\mu\bar{\epsilon}_2 - \frac{i}{2}D_\mu B_3\sigma^\mu\bar{\epsilon}_3 + \frac{i}{2}(D_4 - iD_5)(B_1 + iB_2)\epsilon_3 \\
& - \frac{i}{2}(D_4 - iD_5)B_3\epsilon_2 + \frac{i}{2}D_6(B_1 + iB_2)\epsilon_1 + \frac{i}{2}D_6B_3\epsilon_4 \\
& - \frac{1}{2}[B_1, B_2]\epsilon_4 + \frac{i}{2}[B_1 + iB_2, B_3]\epsilon_1.
\end{aligned} \tag{C.2d}$$

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